Parameterized Quantum Query Algorithms for Graph Problems

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ESA 2024 @Egham Sep 4, 2024

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vertex cover and matching

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kernelization and augmenting paths

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Query Complexity

Given f as an oracle !

Oracle $\mathcal{O}_f$

$f: \{1, \ldots, N\} \to \{0, 1\}$

Query Complexity

Algorithm

ask $f(i)$

output $f(i)$

property of f

Oracle $\mathcal{O}_f$

$f: \{1, \dots, N\} \to \{0, 1\}$

☞ Query Complexity = # of queries to oracle

Quantum Query Complexity

Quantum Algo

$|i, b\rangle$

$|i, b \oplus f(i)\rangle$

property of f

Quantum Oracle $\mathcal{O}_f$

$f: \{1, \dots, N\} \to \{0, 1\}$

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$\sum_i \alpha_i \, |\boldsymbol{i}, b\rangle$

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$\sum_i \alpha_i \, |i, b \oplus f(\boldsymbol{i})\rangle$

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Grover's Algorithm

Input : Oracle access to $\boldsymbol{f}: \{\mathbf{1}, \ldots, \boldsymbol{N}\} \to \{\mathbf{0}, \mathbf{1}\}$
Output : $i \in \{1, \ldots, N\}$ s.t. $f(i) = 1$

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Classical

$\boldsymbol{\Theta}(\boldsymbol{N})$ queries
with error prob. at most 1/3

Quantum

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Classical

$\boldsymbol{\Theta}(\boldsymbol{N})$ queries
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Quantum

$\boldsymbol{O}(\sqrt{\boldsymbol{N}})$ **queries** with error prob. at most 1/3 [Grover '96]

Lower Bound: $\boldsymbol{\Omega}(\sqrt{\boldsymbol{N}})$
[Bennett-Bernstein-Brassard-Vazirani '97]

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Quantum

Classical-Quantum Separation !

$O(\sqrt{N})$ queries with error prob.
at most 1/3 [Grover '96]

Limitation of Quantum Algo

Lower Bound: $\Omega(\sqrt{N})$
[Bennett-Bernstein-Brassard-Vazirani '97]

Quantum Query Complexity for Graph Problems

<u>Adjacency Matrix Model</u>

Quantum oracle access to $E_M: \{1, \ldots, n\} \times \{1, \ldots, n\} \to \{0, 1\}$

$$\boldsymbol{E_M(u, v) = 1 \Leftrightarrow (u, v) \in E(G)}$$

$n = $ # of vertices

Previous Works on Query Complexity

Even through classical algorithms require $\boldsymbol{\Theta(n^2)}$ queries, …

- Connectivity : $\boldsymbol{\Theta(n^{3/2})}$ [Dürr-Heiligman-Høyer-Mhalla '06]
- Maximum Matching : $\boldsymbol{O(n^{7/4})}$ [Kimmel-Witter '21], $\Omega(n^{3/2})$ [Zhang '04]
- Minimum Cut : $\boldsymbol{\Theta(n^{3/2})}$ [Apers-Lee '21]

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Can we achieve $\boldsymbol{O}(\boldsymbol{n}^{2-\epsilon})$ for other problems
such as Vertex Cover, Hamiltonian Path, and Clique ?

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Consider Parameterized Complexity !

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- k-clique : $\tilde{O}(n^{2-2/k})$ [Magniez-Santha-Szegedy '05]

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😋 k : constant
😰 k : large (e.g., $k = \Theta(\log n), k = \Theta(\sqrt{n})$)

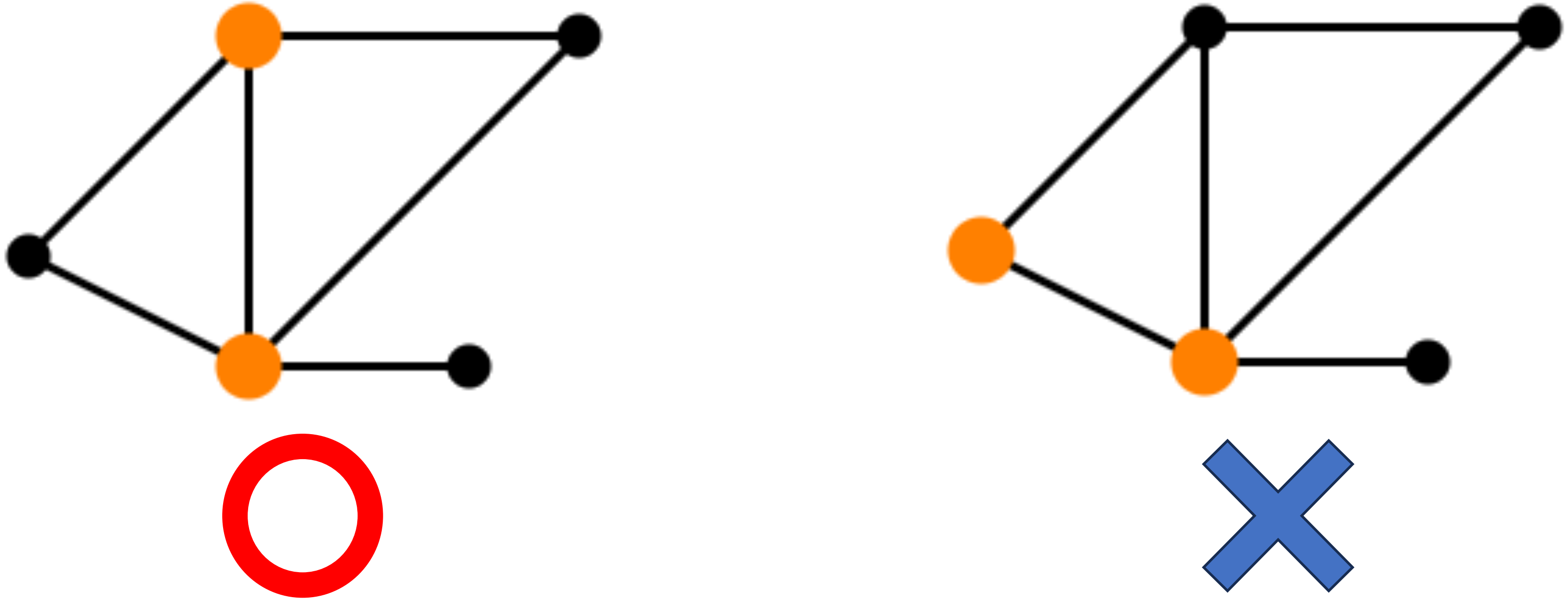
n = # of vertices

k-vertex cover problem

Input : an undirected graph G and an interger k
Find : a **vertex cover** $S \subseteq V$ of size **at most** k

every edge of G has at least one endpoint in S

e.g.)



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	unparameterized	parameterized
Classical time complexity	NP-hard	FPT
		$f(k) \cdot \text{poly}(n)$

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Classical time complexity	NP-hard	FPT
Quantum query complexity	$O(n^2)$ is only known	??

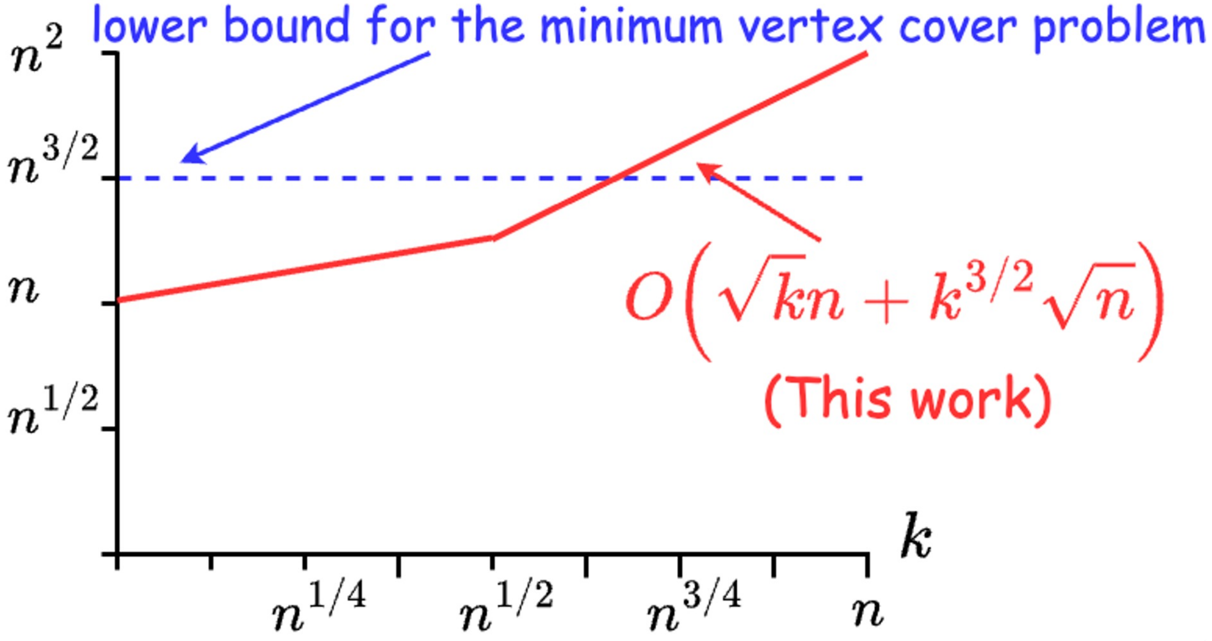
This work: FPT-like quantum query complexity, i.e., $O(f(k) \cdot n^{2-\epsilon})$

Our Contribution 1. Parameterized Quantum Query Complexity for Vertex Cover

Thm.
Quantum Query Complexity to find a vertex cover of size at most k

Upper Bound : $O(\sqrt{k}n + k^{3/2}\sqrt{n})$

FPT-like complexity, i.e., $O(f(k) \cdot n^{2-\epsilon})$



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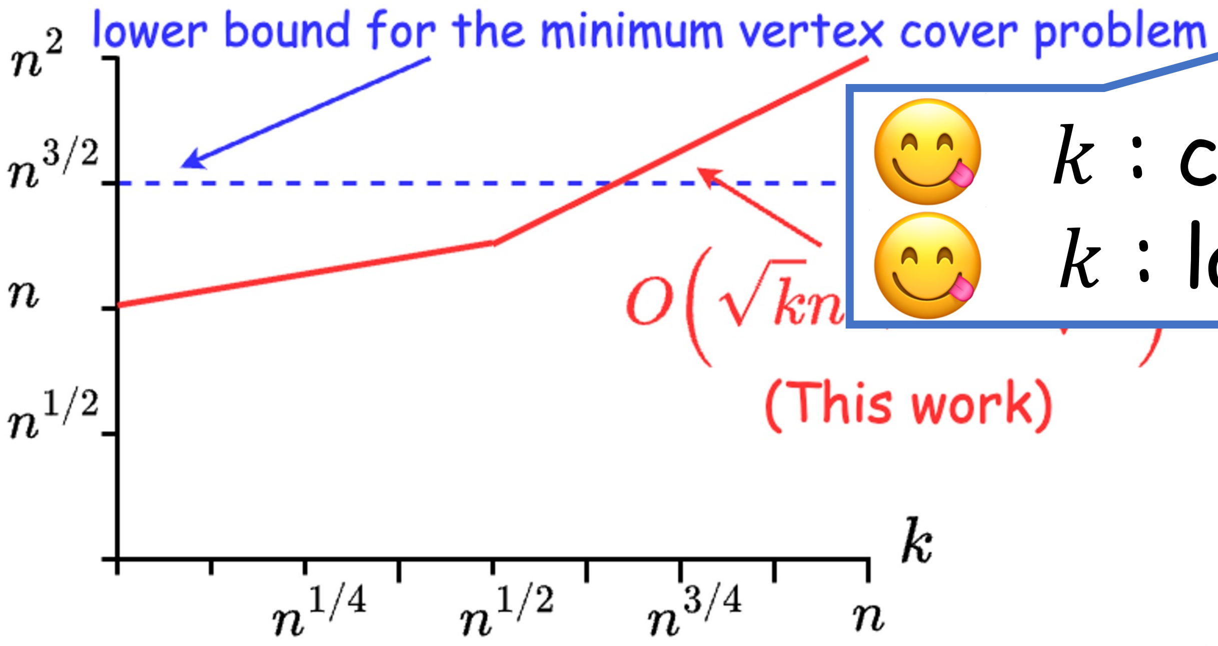

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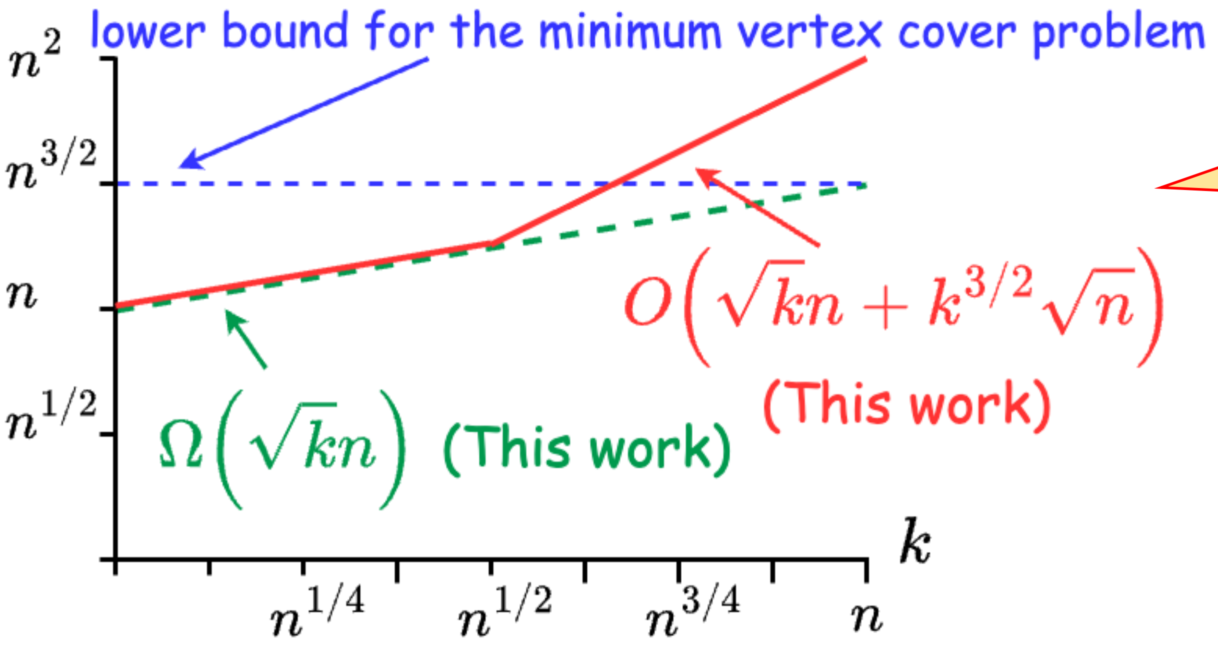
Our Contribution 1. Parameterized Quantum Query Complexity for Vertex Cover

Thm.
Quantum Query Complexity to find a vertex cover of size at most k

Upper Bound : $O(\sqrt{kn} + k^{3/2}\sqrt{n})$
Lower Bound : $\Omega(\sqrt{kn})$ (when $k \le (1-\epsilon)n$)



Optimal complexity
$\Theta(\sqrt{kn})$ when $k = O(\sqrt{n})$

Our Contribution 1. Parameterized Quantum Query Complexity for Vertex Cover

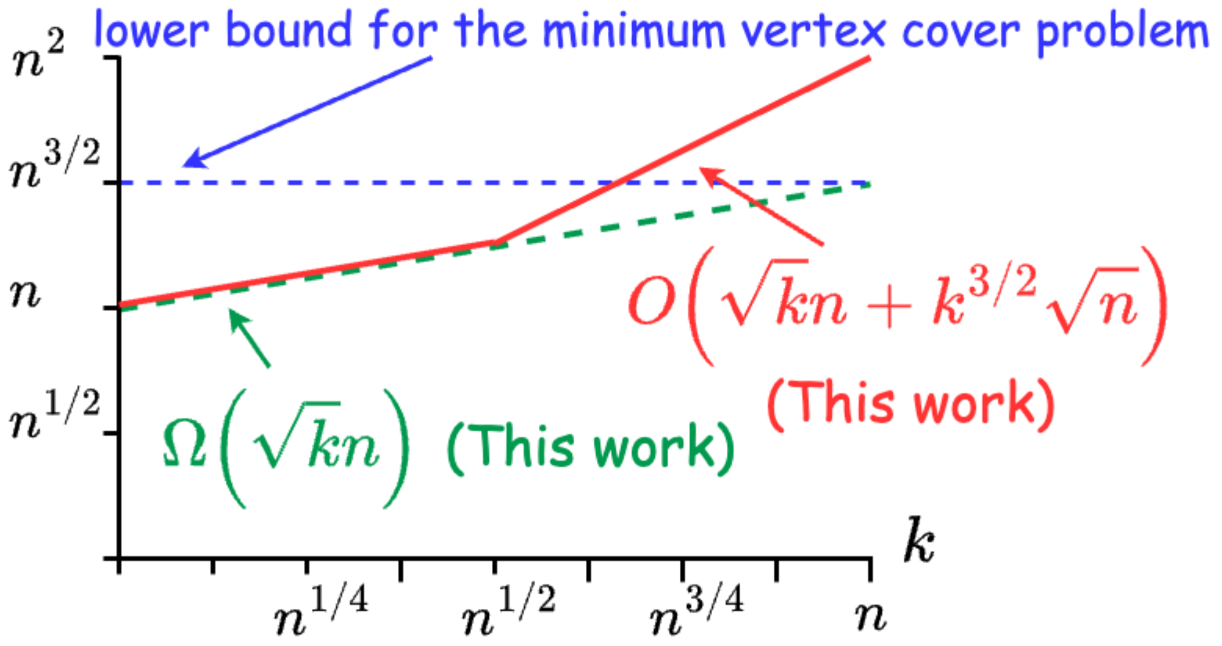
Thm.
Quantum Query Complexity to find a vertex cover of size at most k

Upper Bound : $O(\sqrt{kn} + k^{3/2}\sqrt{n})$

Lower Bound : $\Omega(\sqrt{kn})$ (when $k \le (1-\epsilon)n$)

Significance
- UB $O(n^2)$, LB $\Omega(n^{3/2})$ [Zhang '04] were only known for minimum vertex cover
- Consider Parameterized ver.

Technique
- Quantum Query Kernelization



Kernelization

kernel

Input : instance (G, k)
Output : another equivalent small instance (G', k'),
or conclude that (G, k) is a Yes-instance or a No-instance

- (G, k) is a Yes instance $\Leftrightarrow$ (G', k') is a Yes instance
- $E(G') \leq f(k)$
- $k' \leq g(k)$

Buss-Goldsmith's Kernelization for k-vertex cover

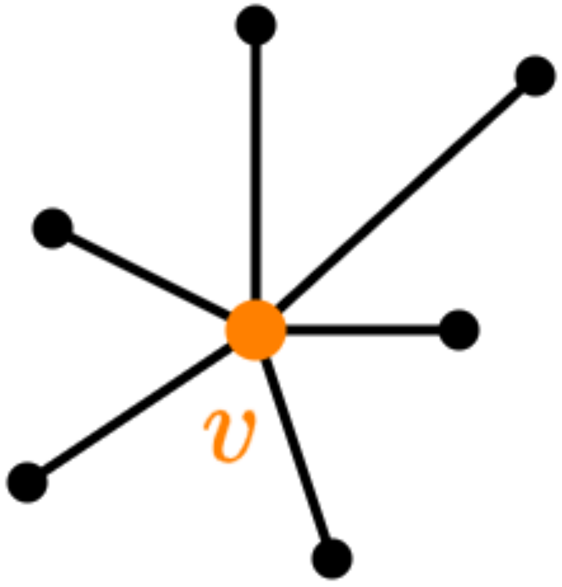
Rule 1. If G has an isolated vertex v, then $\boldsymbol{(G, k) \rightarrow (G - v, k)}$

Buss-Goldsmith's Kernelization for k-vertex cover

Rule 1. If G has an isolated vertex v, then $(G, k) \rightarrow (G - v, k)$

Rule 2. If G has a vertex v of **degree at least $k+1$**, then

If v is not in a vertex cover, then it must contain all neighbors of v.



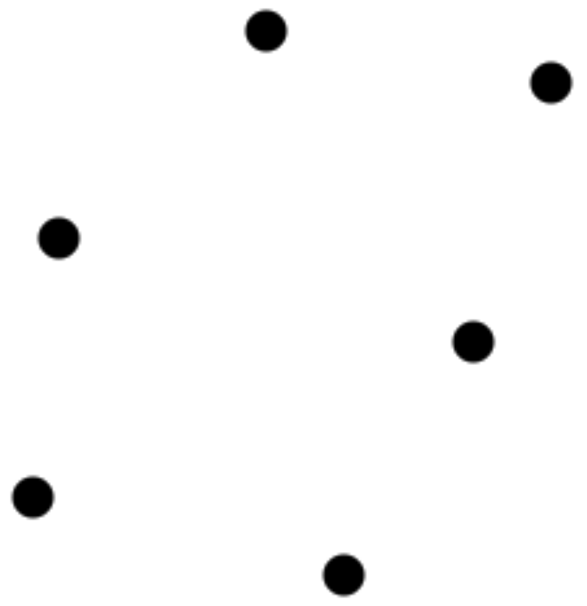
Buss-Goldsmith's Kernelization for k-vertex cover

Rule 1. If G has an isolated vertex v, then $(G, k) \rightarrow (G - v, k)$

Rule 2. If G has a vertex v of degree at least $k+1$, then

$$\boldsymbol{(G, k) \rightarrow (G - v, k - 1)}$$

v must be in any vertex cover of size at most k.



Buss-Goldsmith's Kernelization for k-vertex cover

Rule 1. If G has an isolated vertex v, then $(G, k) \to (G - v, k)$

Rule 2. If G has a vertex v of degree at least $k + 1$, then
$(G, k) \to (G - v, k - 1)$

v must be in any vertex cover of size at most k.

Fact: After Applying Rules 1 and 2, if $|E(G)| > k^2$,
then (G, k) is a No instance

New Approach: Quantum Query Kernelization

Input : Oracle access to $(\boldsymbol{G}, \boldsymbol{k})$

Output : another **equivalent** instance $(\boldsymbol{G}', \boldsymbol{k}')$ **as a bit string,**
or conclude that (G, k) is a Yes-instance or a No-instance

- $(\boldsymbol{G}, \boldsymbol{k})$ is a Yes instance $\Leftrightarrow$ $(\boldsymbol{G}', \boldsymbol{k}')$ is a Yes instance

New Approach: Quantum Query Kernelization

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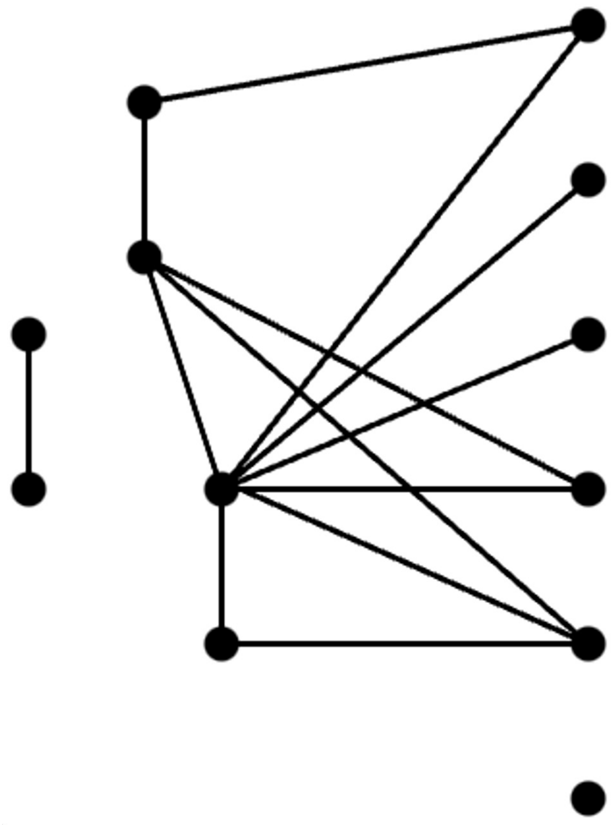
- $(\boldsymbol{G}, \boldsymbol{k})$ **is a Yes instance** $\Leftrightarrow$ $(\boldsymbol{G}', \boldsymbol{k}')$ **is a Yes instance**

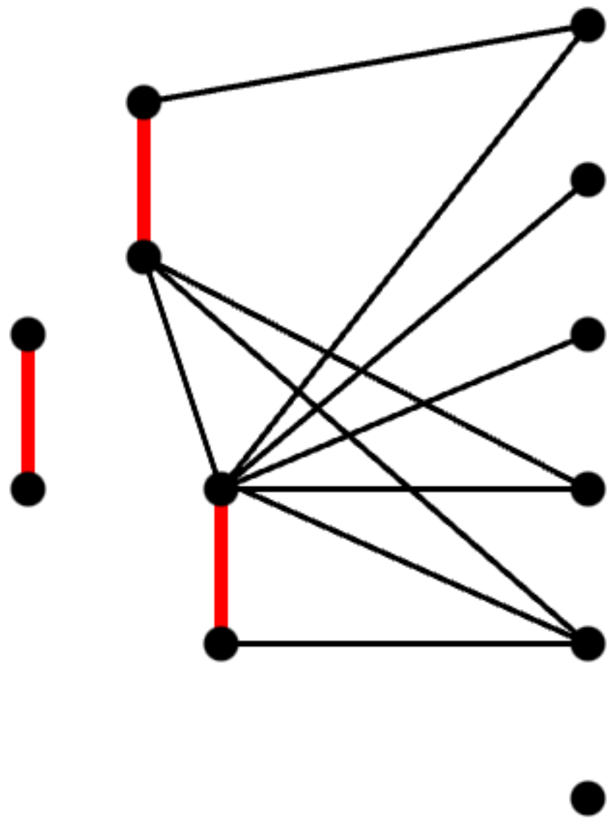
☞ **After Applying quantum query kernelization,**
just apply classical algorithm for $(\boldsymbol{G}', \boldsymbol{k}')$.

Classical Kernelization Suitable for Quantum Algo



Classical Kernelization Suitable for Quantum Algo

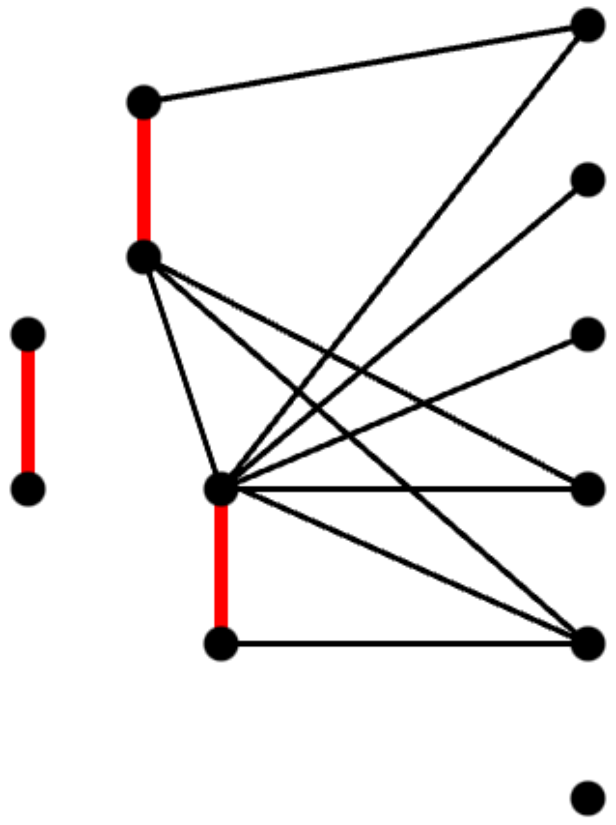
Step1 Find a **maximal matching** M



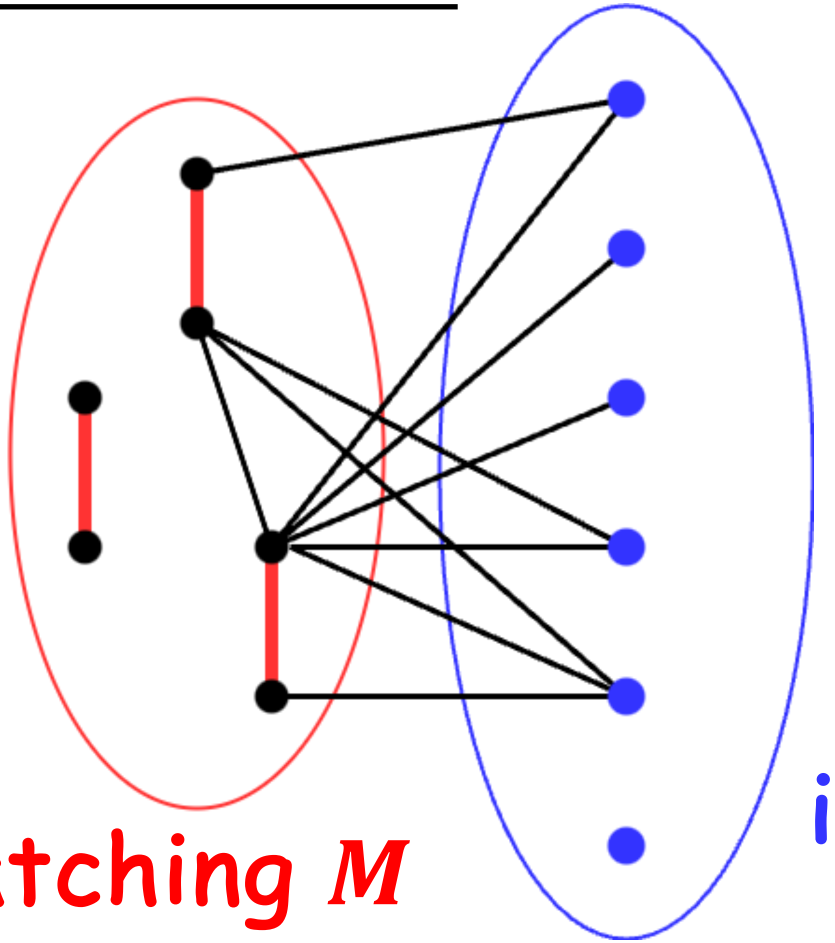
Classical Kernelization Suitable for Quantum Algo

Step1 Find a maximal matching M

if $|M| > k$: then No instance



Crucial Observation



maximal matching M

independent set

☞ All edges touch an endpoint of an edge in M !

Classical Kernelization Suitable for Quantum Algo

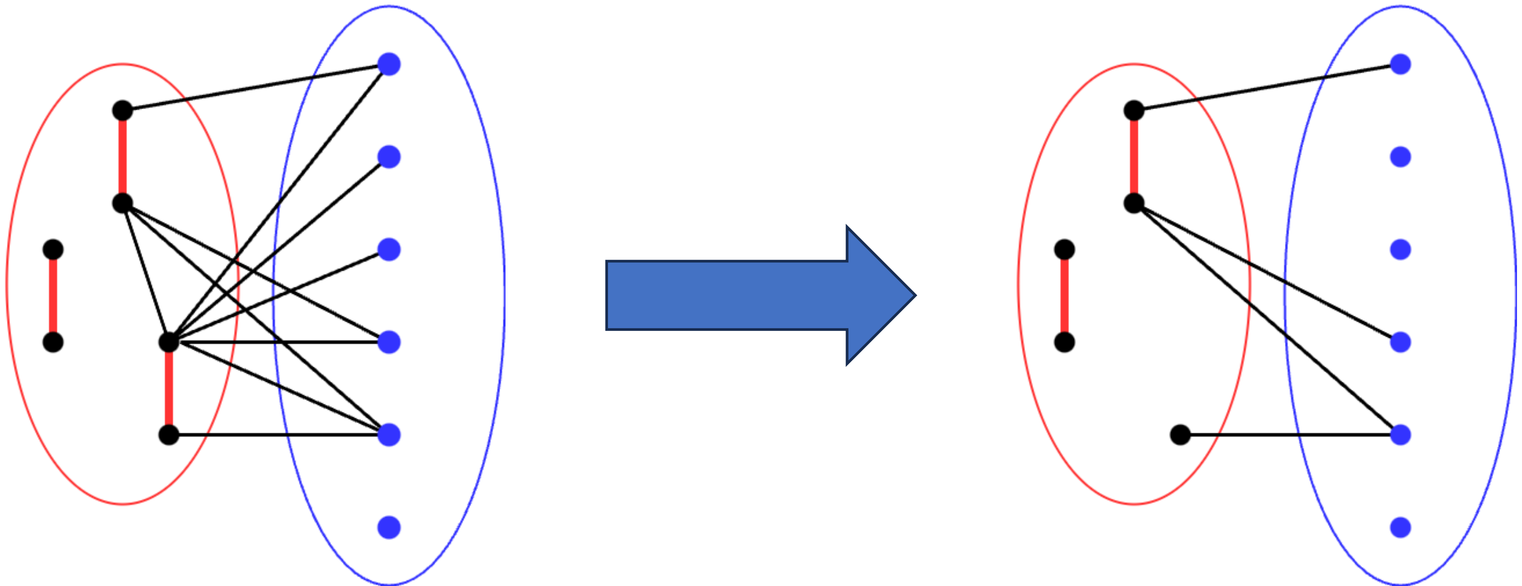
Step1 Find a maximal matching M

if $|M| > k$: then No instance

Step2 Apply Rule 2 only for endpoints of edges in M

Rule 2. If G has a vertex v of degree at least $k+1$, then

$$(G, k) \rightarrow (G - v, k-1)$$



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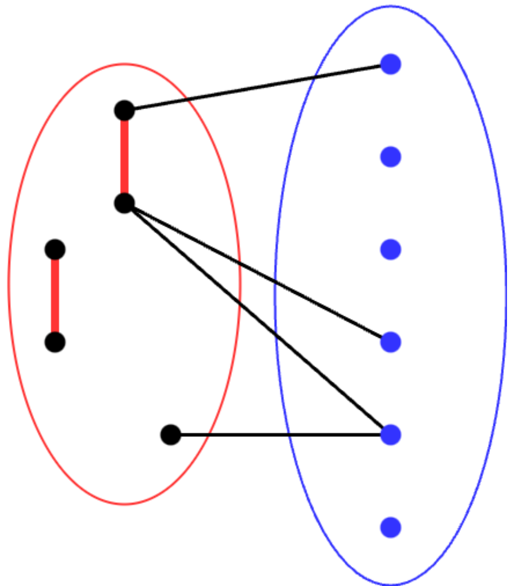


Lem: After Step1 and 2, $|E(G)| \leq 2k^2$

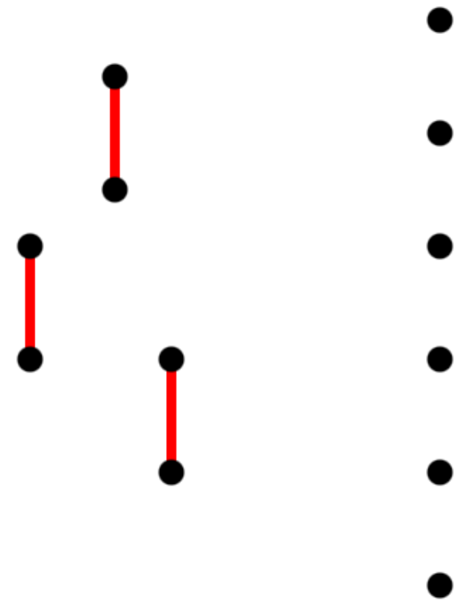
Quantum Query Kernelization

Step1 Find a matching of size at least $k+1$

or

a maximal matching M of size at most k

using Grover's search



Lem: Step1 uses $O(\sqrt{kn})$ queries

Quantum Query Kernelization

Step1 Find a matching of size at least $k+1$ or a maximal matching M of

No instance

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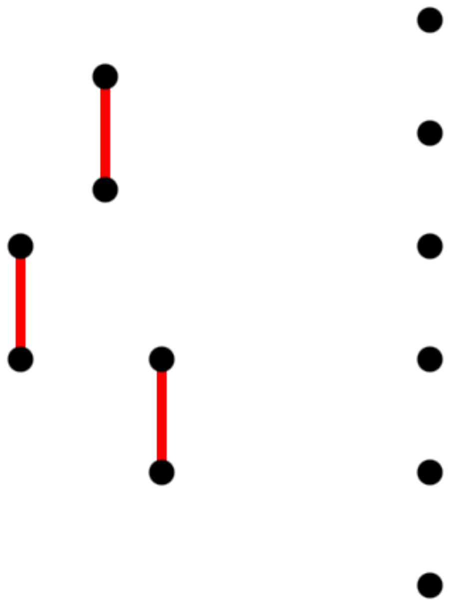

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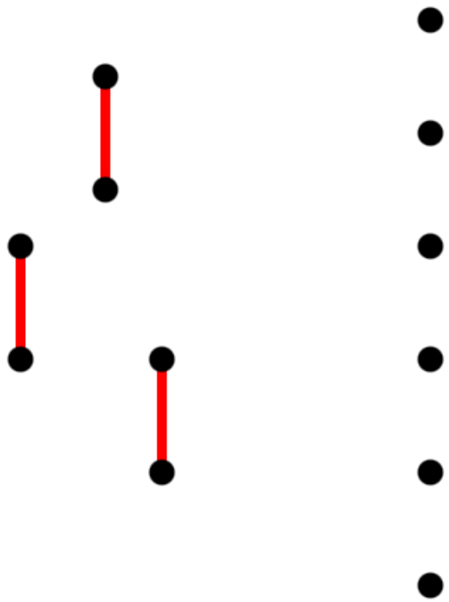

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a matching of size at least $k+1$

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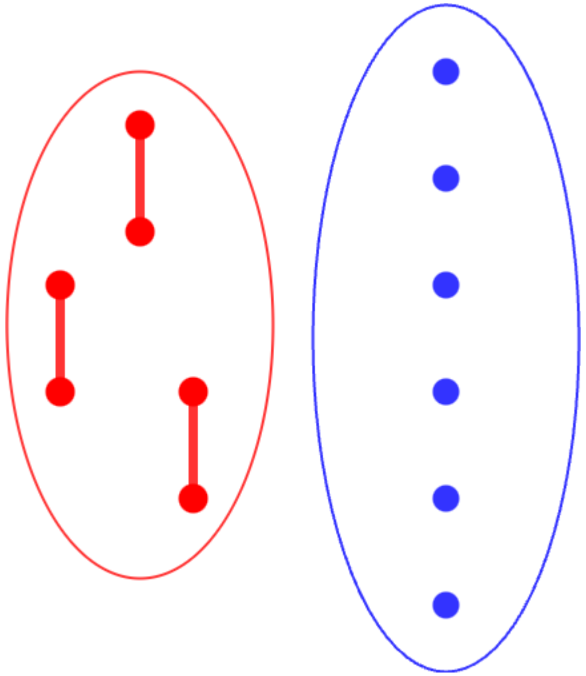
Step2 For each $v \in V(M)$:

all endpoints of edges in M



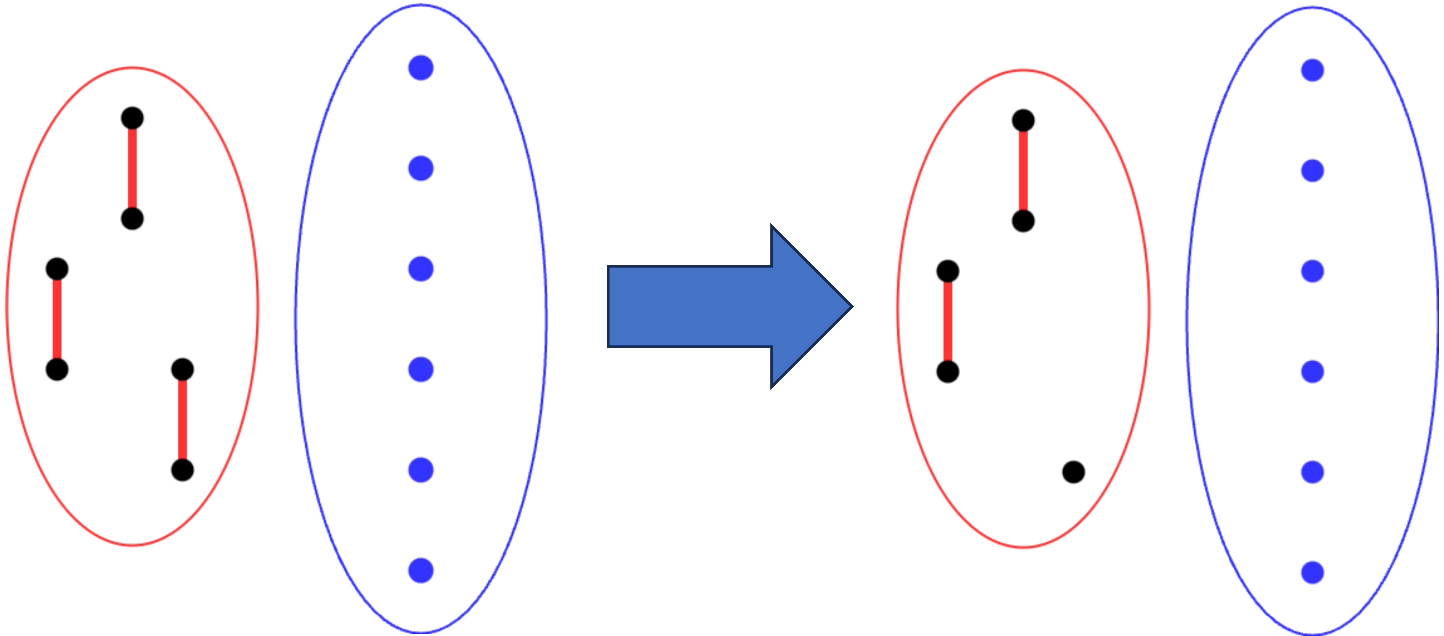
Quantum Query Kernelization

Step1 Find a matching of size at least $k+1$
or
a maximal matching M of size at most k

Step2 For each $v \in V(M)$: all endpoints of edges in M

if (degree of v) $> k$: then remove v, $k \leftarrow k - 1$

else: find all edges incident to v using Grover's search



Quantum Query Kernelization

a matching of size at least $k+1$

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or

a maximal matching $\boldsymbol{M}$ of size at most k

Step2 For each $\boldsymbol{v} \in \boldsymbol{V}(\boldsymbol{M})$:

all endpoints of edges in $\boldsymbol{M}$

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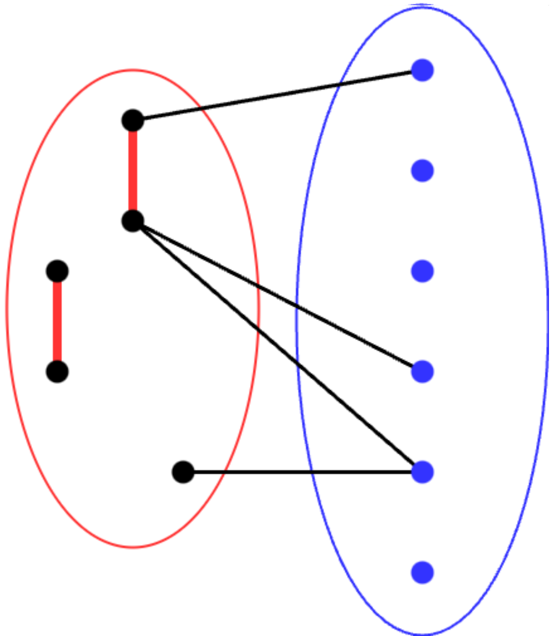

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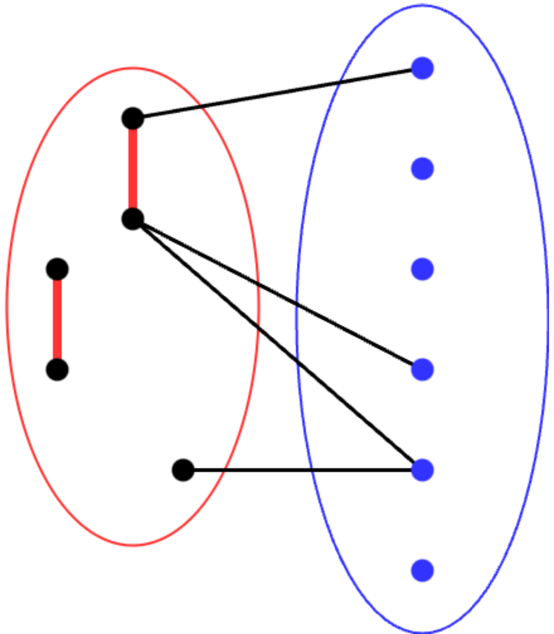

Obtain an equivalent instance as a bit string !

Quantum Query Kernelization

a matching of size at least $k+1$

Step1 Find

or

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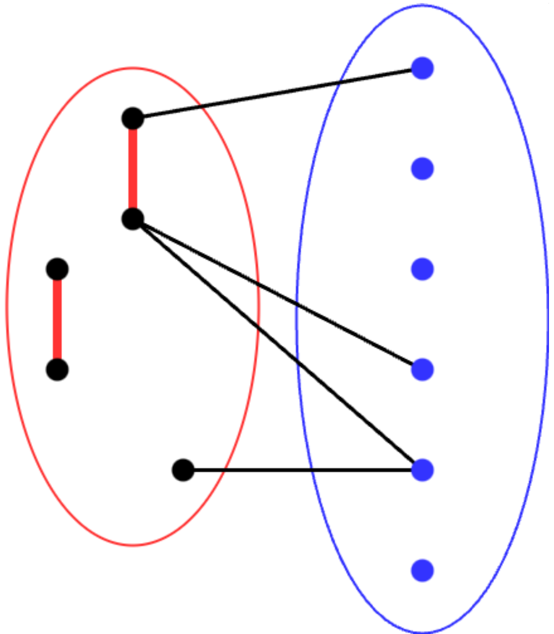


Lem:

Step2 uses $\boldsymbol{O(k^{3/2}\sqrt{n})}$ queries

Quantum Query Kernelization

<u>Step1</u> Find

a matching of size at least $k + 1$

or

a maximal matching $\boldsymbol{M}$ of size at most k

using Grover's search

$\Leftarrow$ $\boldsymbol{O(\sqrt{kn})}$ queries

<u>Step2</u> For each $\boldsymbol{v} \in \boldsymbol{V(M)}$:

if (degree of v) $> k$: then remove v, $k \leftarrow k - 1$

else: find all edges incident to v using Grover's search

$\Leftarrow$ $\boldsymbol{O(k^{3/2}\sqrt{n})}$ queries

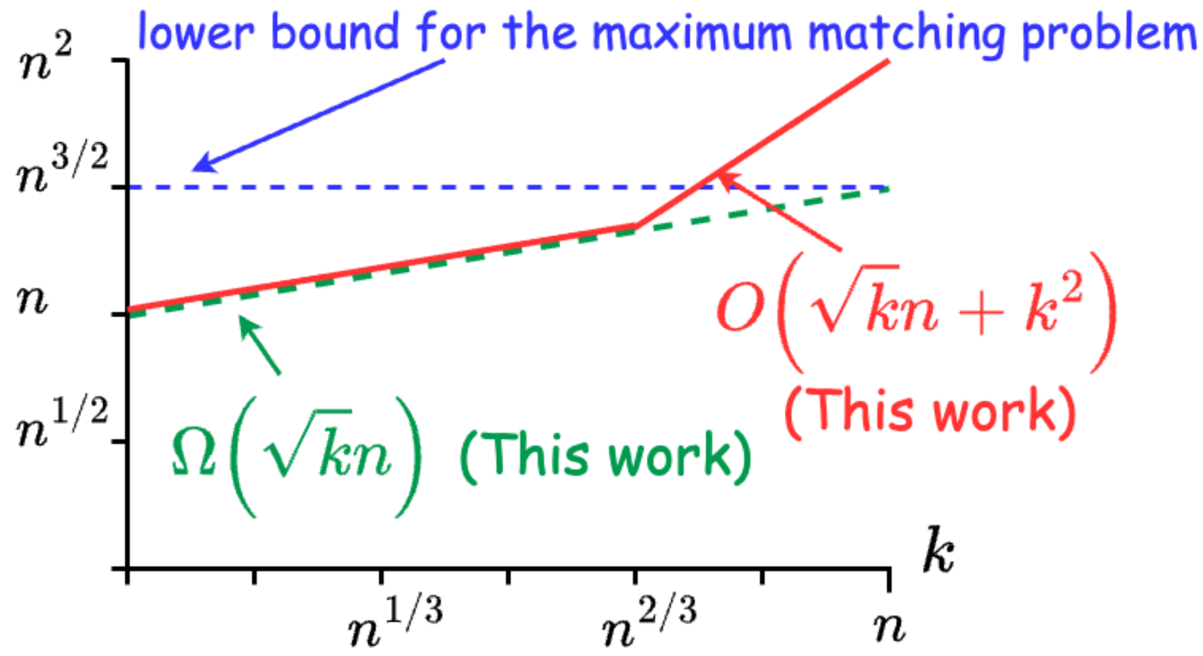
Our Contribution 2. Parameterized Quantum Query Complexity for Matching

Thm.

Quantum Query Complexity to find a matching of size at least k

Upper Bound : $O(\sqrt{kn} + k^2)$

Lower Bound : $\Omega(\sqrt{kn})$



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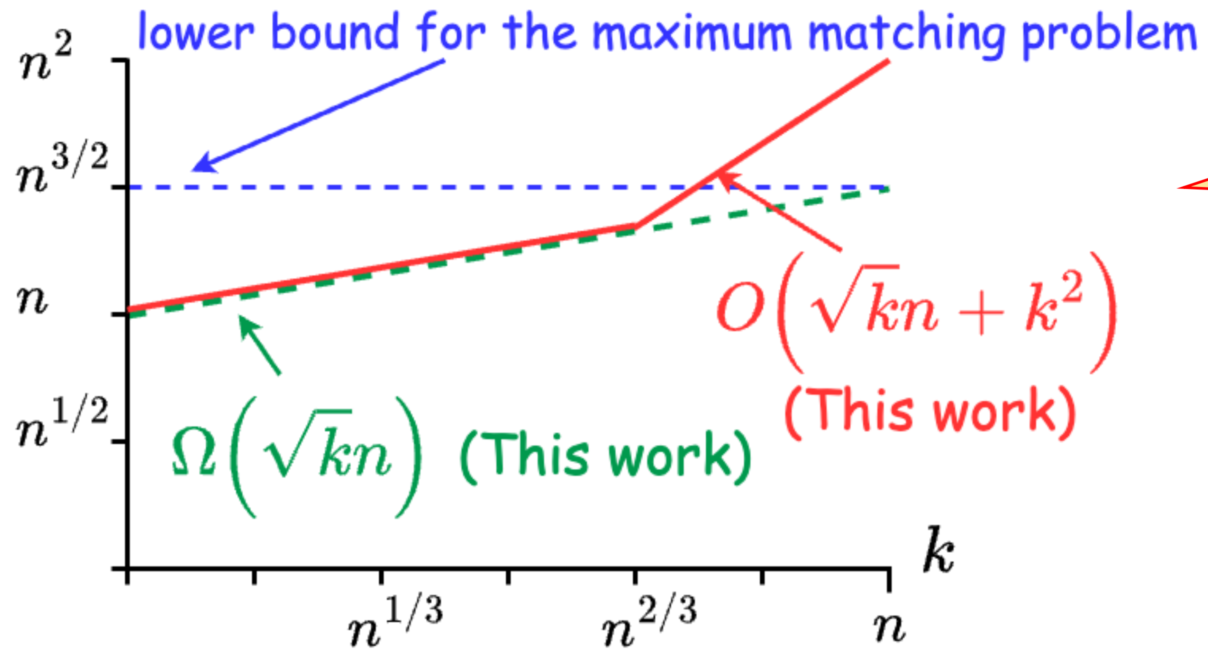

Optimal complexity
$\Theta(\sqrt{kn})$ when $k = O(n^{2/3})$

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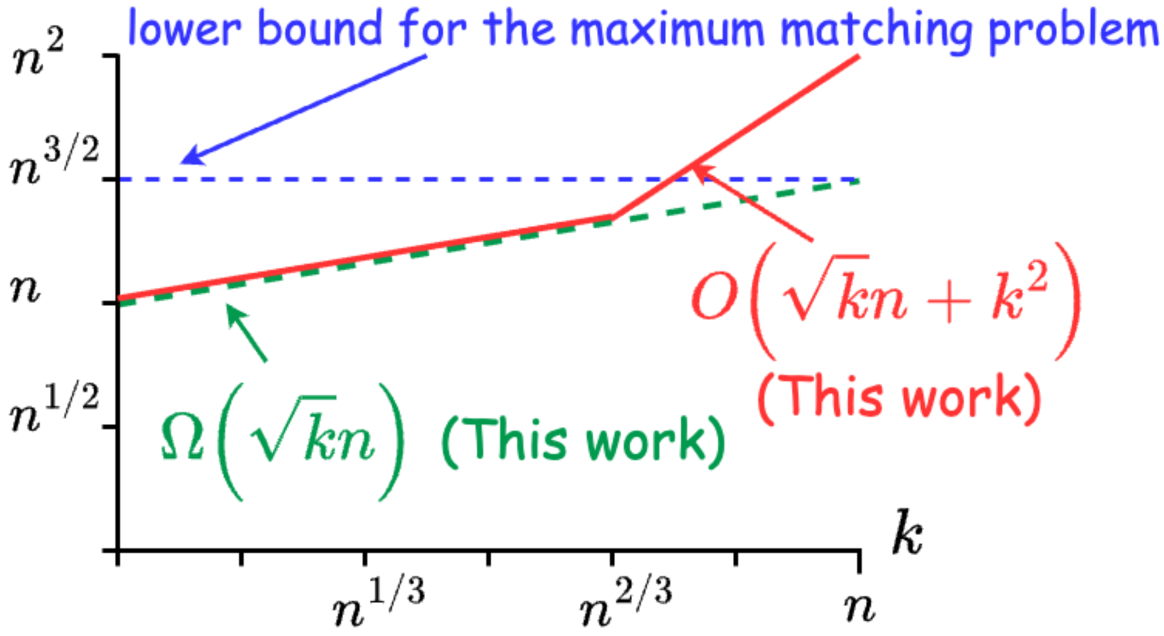
- Consider **Parameterized** ver.

Technique

- augmenting paths
- quantum query kernelization idea



Conclusion

- Consider Parameterized Quantum Query Complexities
- Obtain Optimal Parameterized Quantum Query Complexities for vertex cover and matching when the parameters are not so large.

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- Obtain Optimal Parameterized Quantum Query Complexities for vertex cover and matching when the parameters are not so large.

Message

☞ By making smart use of classical techniques such as kernelization, we can improve quantum query complexities !

Appendix

Quantum Query Algo for k-matching

a matching of size at least $k+1$

Step1 Find or ⟸ $O(\sqrt{kn})$ queries

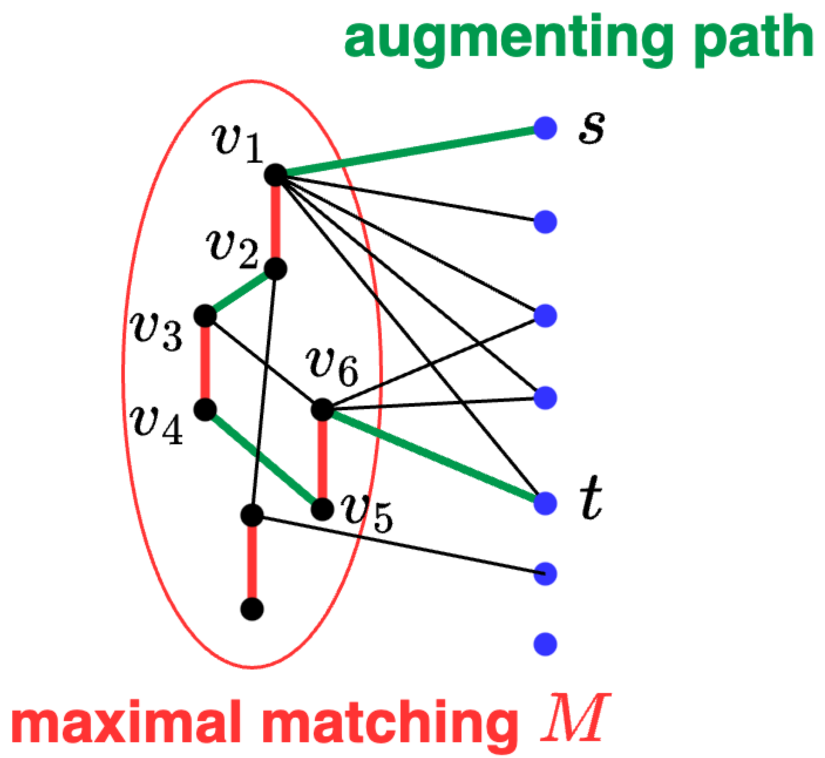
a maximal matching of size at most k

Quantum Query Algo for k-matching

Step1 Find a matching of size at least $k+1$ or a maximal matching M of size at most k ← $O(\sqrt{kn})$ queries

Step2 Repeatedly find an augmenting path and augment along it

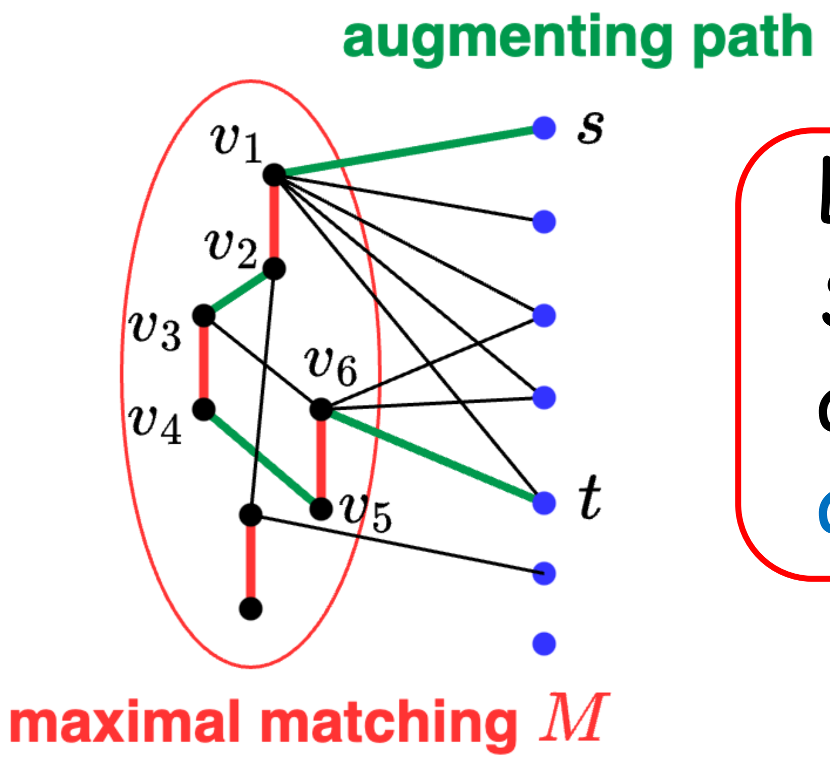
$|M|$ increases by 1 !



Quantum Query Algo for k-matching

Step1 Find a matching of size at least $k+1$ or a maximal matching M of size at most k ⇦ $O(\sqrt{kn})$ queries

Step2 Repeatedly find an augmenting path and augment along it

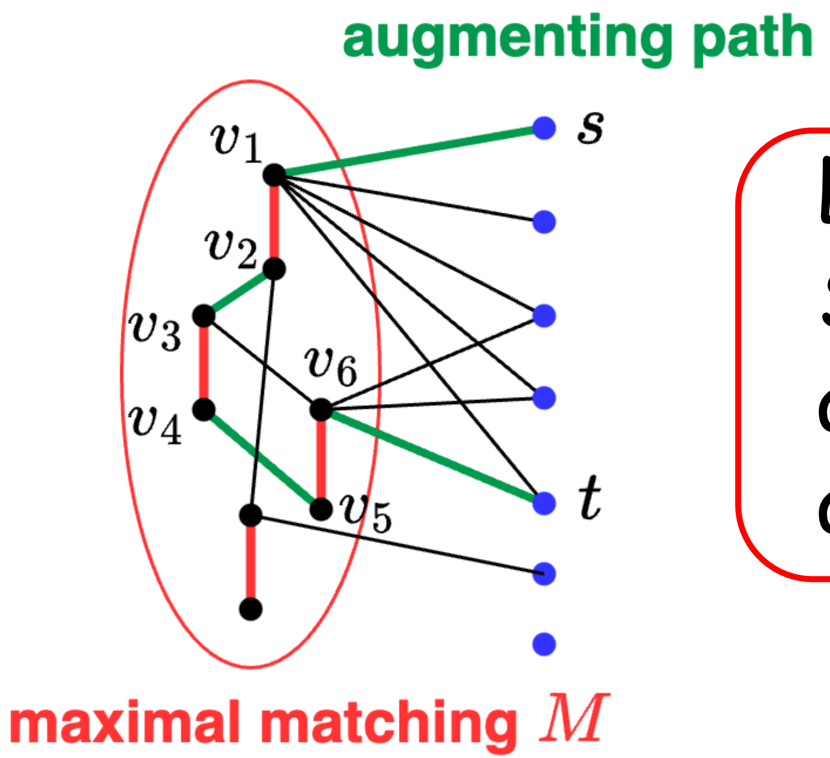


Lem:
Step2 uses $O(k^2)$ queries + amoritized $O(\sqrt{n})$ queries per one augmentation

Quantum Query Algo for k-matching

Step1 Find a matching of size at least $k+1$ or a maximal matching M of size at most k ⟸ $O(\sqrt{kn})$ queries

Step2 Repeatedly find an augmenting path and augment along it



Lem:
Step2 uses $O(k^2)$ queries + amoritized $O(\sqrt{n})$ queries per one augmentation

$O(k^2 + k\sqrt{n})$ queries